Wire Dipoles: DC approximation

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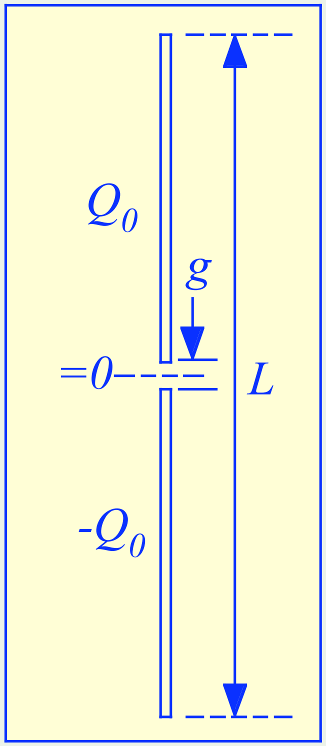
Background: Simple Idea

- The idea here is to treat a wire dipole as a near-DC system:
  - Linearly decreasing current from the (central) feed to the wire ends
  - Wire capacitance arises from constant charge on each wire
  - Wire inductance calculated from the DC model (but accounting roughly for the non-constant current)
- The dipole will present a real impedance when the inductive and capacitive reactances cancel (resonant condition)

Wire Dipole Capacitance

- Consider a wire dipole of length L excited at very low frequency
- If the length of the dipole is << the wavelength, current falls approximately linearly with distance from the center => constant charge density along the wire
- What is the potential between the two wires for a known charge (and thus current / frequency)?



Find potential along wire axis

- Let   = position along rod
- potential = integral of charge in the usual fashion

  In order to keep all the log arguments positive as they should be, split integral into [opposite wire] + [less than] + [greater than]; shown here is the case of >0

$$(\ ) = \frac{{}_0 c^2}{4}\left(\int_{L/2}^{g/2} \frac{Q_0}{\sqrt{(z\ \ )^2+a^2}}dz + \int_{g/2} \frac{Q_0}{\sqrt{(\ \ z)^2+a^2}}dz + \int^{L/2}\frac{Q_0}{\sqrt{(z\ \ )^2+a^2}}dz \right)$$

$$= \frac{Q_0\ {}_0c^2}{4}\left(\int_{+L/2}^{+g/2}\frac{1}{\sqrt{\ ^2+a^2}}d\ \ + \int_{g/2}^{0}\frac{1}{\sqrt{\ ^2+a^2}}d\ \ + \int_0^{L/2}\frac{1}{\sqrt{\ ^2+a^2}}d\ \ \right)$$

$$= \frac{Q_0\ {}_0c^2}{4}\left(\ln\frac{\ +g/2+\sqrt{(\ +g/2)^2+a^2}}{\ +L/2+\sqrt{(\ +L/2)^2+a^2}} \quad \ln\frac{a}{g/2+\sqrt{(\ \ g/2)^2+a^2}} + \ln\frac{L/2\ \ +\sqrt{(L/2\ \ )^2+a^2}}{a}\right)$$

$$\sqrt{(\ +g/2)^2+a^2} \qquad (\ +g/2)+\frac{a^2}{2(\ +g/2)}$$

$$\frac{Q_0\ {}_0c^2}{4}\left(\ln\frac{2\ +g+\frac{a^2}{2\ +g}}{L+2\ +\frac{a^2}{2\ +L}} + \ln\frac{\left(g/2+\sqrt{(\ \ g/2)^2+a^2}\right)\left(L/2\ \ +\sqrt{(L/2\ \ )^2+a^2}\right)}{a^2}\right)$$

Approximate radicals

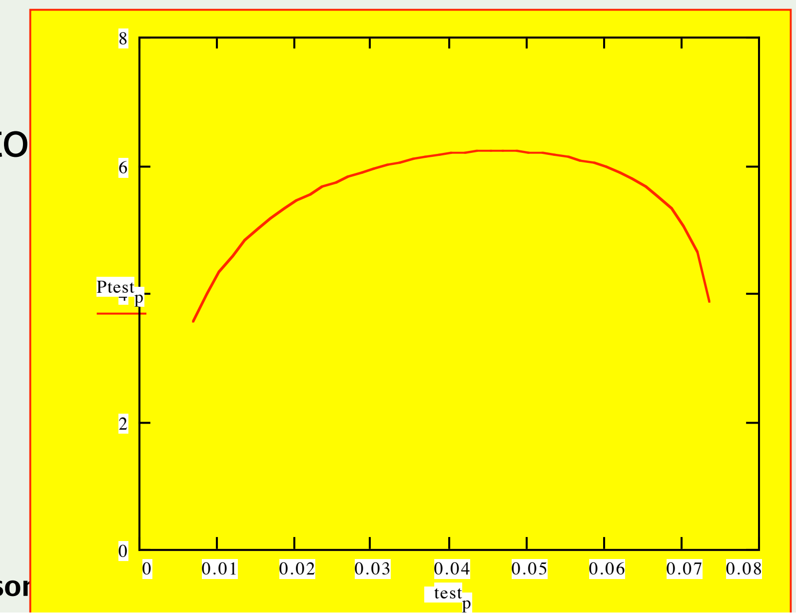
expand radicals in second bracket (okay except near ends)

$$\frac{Q_0 \quad _0 c^2}{4} \ln \left(\frac{2 \quad + g + \frac{a^2}{2 \quad + g}}{L + 2 \quad + \frac{a^2}{2 \quad + L}} \right) + \ln \left(\frac{\left(2 \quad g + \frac{a^2}{2 \quad g}\right)\left(L \quad 2 \quad + \frac{a^2}{L \quad 2}\right)}{a^2} \right)$$

$$\frac{Q_0 \quad _0 c^2}{4} \ln \left(\frac{(2 \quad + g)}{(L + 2 \quad)} \right) + \ln \left(\frac{(2 \quad g)(L \quad 2 \quad)}{a^2} \right)$$

$$= \frac{Q_0 \quad _0 c^2}{4} \ln \left(\frac{(2 \quad + g)}{(L + 2 \quad)} \frac{(2 \quad g)(L \quad 2 \quad)}{a^2} \right) = \frac{Q_0 \quad _0 c^2}{4} \ln \left(\frac{(4 \quad ^2 \quad g^2)}{a^2} \frac{(L \quad 2 \quad)}{(L + 2 \quad)} \right)$$

- Potential is hardly constant along the wire [the charge density would need to increase at the ends] but it’s okay for an estimate...



Potential and Capacitance

- The applied voltage is approximately the difference in potential between the centers of the two arms of the antenna ($=L/4$):

$$V_a \quad \frac{2Q_0 \quad {}_0 c^2}{4} \ln \frac{4\left(\frac{L}{4}\right)^2 \quad g^2}{a^2} \frac{L \quad 2\frac{L}{4}}{L+2\frac{L}{4}} = \frac{2Q_0 \quad {}_0 c^2}{4} \ln \frac{\frac{L^2}{4} \quad g^2}{a^2} \frac{\frac{L}{2}}{\frac{3L}{2}}$$

$$\frac{2Q_0 \quad {}_0 c^2}{4} \ln \frac{1}{12} \frac{L}{a}^2 = \frac{2Q_0 \quad {}_0 c^2}{4} + 2\ln \frac{L}{a} \quad \ln(12)^{-} = \frac{Q_0 \quad {}_0 c^2}{} + \ln \frac{2L}{a} \quad \ln(2\sqrt{12})^{-}$$

- The capacitance is then C=Q/V; the charge on each plate is Q = Q_0L/2

$$C = \frac{Q}{V_a} = \frac{Q_0 \frac{L}{2}}{\frac{Q_0 \quad {}_0 c^2}{} \ln \frac{2L}{a} \quad \ln(2\sqrt{12})_{-}} = \frac{L}{2 \quad {}_0 c^2 \ln \frac{2L}{a} \quad \ln(2\sqrt{12})_{-}}$$

Resonance

- This capacitance can be regarded as in series with the inductance of the wires; the resonant frequency is inversely proportional to the product of inductance and capacitance
- however, since the charge is uniformly distributed over the wire we ought roughly to use an effective length of half the actual length of the wire…we find:

Inductance of wire (from "Potentials and Inductance"): $= \frac{{}_0(L/2)}{2}\left[\ln\left(\frac{L}{a}\right)-1\right]$; find resonant frequency...

$$ {}_r^2 = \frac{1}{C} = \frac{1}{\frac{{}_0(L/2)}{2}\left[\ln\left(\frac{L}{a}\right)-1\right]\frac{L}{2\ {}_0c^2\left[\ln\left(\frac{2L}{a}\right)-\ln(2\sqrt{12})\right]}} = \frac{8c^2\left[\ln\left(\frac{2L}{a}\right)-\ln(2\sqrt{12})\right]}{L^2\left[\ln\left(\frac{L}{a}\right)-1\right]} $$

Now multiply both sides by L^2

$$ (2\ \ f_r)^2 L^2 = \frac{8c^2\left[\ln\left(\frac{2L}{a}\right)-\ln(2\sqrt{12})\right]}{\ln\left(\frac{L}{a}\right)-1}; \text{ divide by } f^2 \text{ using } \frac{c}{f} = \qquad (2\ \ )^2 L^2 = \frac{8\ \ ^2\left[\ln\left(\frac{2L}{a}\right)-\ln(2\sqrt{12})\right]}{\ln\left(\frac{L}{a}\right)-1} $$

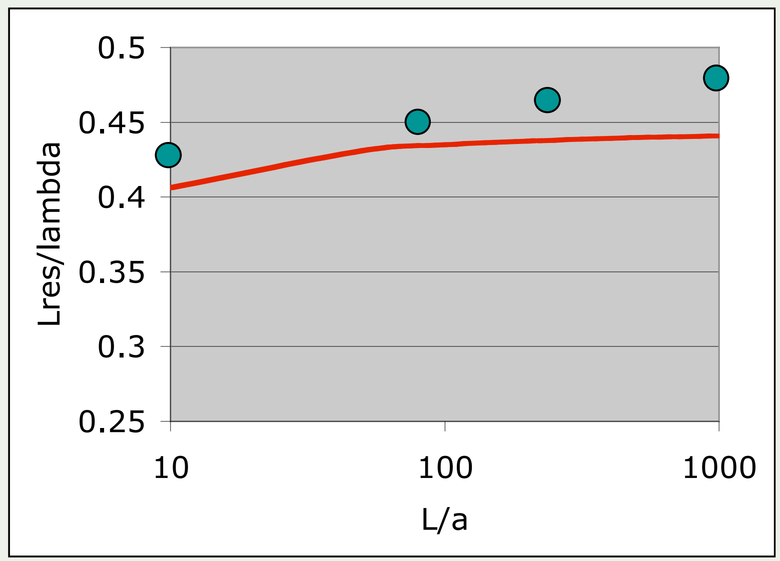
Resonant Length

- Now divide both sides by the square of the wavelength to arrive at an expression for the normalized L at which the antenna is resonant
  - note this is an implicit expression if we're solving for length of a fixed diameter wire, but explicit if we allow the radius of the wire a to vary while keeping the ratio L/a fixed.

$$\frac{L}{\ }^{2} = \frac{2 \quad \ln \frac{2L}{a} \qquad \ln(2\sqrt{12})\_}{{}^{2} \quad \ln \frac{L}{a} \qquad 1\_}$$

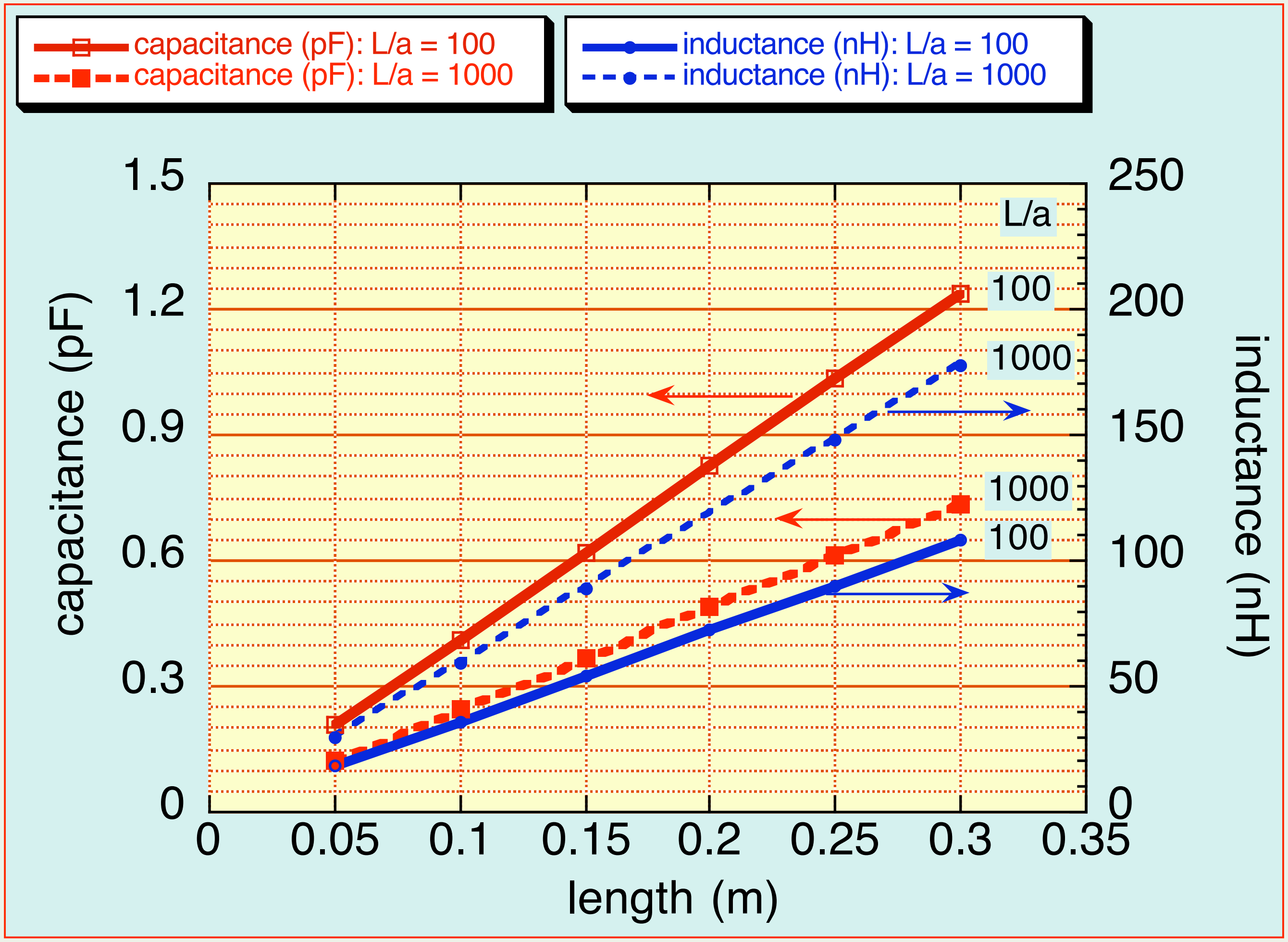
$$\frac{L_{res}}{\ } = \frac{\sqrt{2}}{\ } \sqrt{\frac{\ln \frac{2L}{a} \qquad \ln(2\sqrt{12})\_}{\ln \frac{2L}{a} \qquad \ln 2 \quad 1\_}}$$

- Estimate is close to the 'right' answer (points from Jenn, Naval Postgraduate School, Stutzman & Thiele [1st ed] p. 200, 203), though effect of L/a appears to be underestimated at large values
- Note for arbitrarily small wire diameter, the estimate approaches 0.45, which is about 10% below the correct answer of 0.5



What’s Going On...

- Both capacitance and inductance scale linearly in length for fixed (L/a)
- Making the wire thinner makes the inductance go up and the capacitance go down; the effects nearly cancel since the resonant frequency goes as the square root of (1/ C); thus the resonant frequency is only a weak function of the wire diameter...



Discussion

- The input impedance of a half-wave dipole can thus be understood approximately as composed of the low-frequency capacitance of the dipole in series with its low-frequency inductance
- The estimate is not exact, nor should it be:
  - the actual half-wave dipole current is nearly sinusoidal, not linear
  - the capacitance was estimated using a fixed charge density, whereas we should properly correct the charge to get a fixed potential along the wire
  - the inductance correction (1/2) was a wild guess rather than a careful estimate of the distributed equivalent inductance
- Effects of variant configurations are clear:
  - tip-loading (bump on end of antenna) adds capacitance resonating with whole wire length
  - inductor at feed increases inductance resonating all capacitance
  - both act to shorten antenna for the same resonant frequency